NOTES FROM NCNGT JOINT PROBLEM SESSION

As part of the NCNGT conference, we (Jonathan Johnson, Siddhi Krishna, Maggie Miller, Jung Park, Matt Stoffregen, Linh Truong, and Hannah Turner) hosted a (virtual) joint problem session on June 9, 2020. This is the write–up of that problem session, where we discussed questions related to the *Investigating the L–space Conjecture*, *Knots, Surfaces, and 4–Manifolds*, and *Floer homology and low–dimensional topology* topic groups. We thank Matt Hedden and Tye Lidman for generously helping us moderate this session.

1. Investigating the L–space conjecture

Question (Francesco Lin): *How big can a hyperbolic L–space be?*

There are examples of hyperbolic L–spaces with volume getting arbitrarily large (for example, let K be the figure–8 knot, and let $\Sigma_n(K)$ denote the n–fold cyclic branched cover over K. These manifolds are all L–spaces, and as $n \to \infty, Vol(\Sigma_n(K)) \to \infty$). There are also examples of hyperbolic L–space knots with volume getting arbitrarily large (see [Bak08]), so large surgeries on them will also provide examples.

Here are some related questions, in order of difficulty:

- (Hardest question) *Does there exist an infinite family of L-spaces such that the injectivity radius goes to infinity?* Note: there are examples of rational homology spheres for which the injectivity radius gets arbitrarily large; this is due to Calegari–Dunfield [CD06], and it is a deep construction using number theory.
- (Slightly easier, but still hard) A family of hyperbolic manifolds Y_n **converges to** $\mathbb{H}^3$ **in the Benjamini–Schramm (BS) sense** if for all r, $Vol(Y_{n,r})/Vol(Y_n) \longrightarrow 0$ for any fixed r, where $Y_{n\leq r}$ denotes the subset of Y_n with injectivity radius less than r (see [BD15]). (That is, the manifold has large injectivity radius everywhere.) *Do there exist a family of L–spaces converging to* $\mathbb{H}^3$ *in the BS sense?*
- *Does there exist a family of hyperbolic 3–manifolds, together with a sequence of points embedded therein, such that the injectivity radius at the points goes to infinity?*

Francesco guesses that the answer for these three questions should be "no" (because from the point of view of spectral geometry, they should have small λ_1 for Δ).

NB: You could vary this question based on the existence of orderability or taut foliations.

Question (John Baldwin): *Prove that for all $r \in (-1, 1)$, $S^3_r(K)$ has a taut foliation for K non–fibered.*

In the early 2000's, Rachel Roberts wrote papers on when surgeries on fibered knots admit taut foliations [Rob01a, Rob01b]. She proved a quantitative result: if K is fibered with pseudo–Anosov monodromy, then (with the right coordinate system) there exists (at least) a half–infinite interval worth of surgeries along K with taut foliations.

Rachel Roberts and Tao Li proved that for any non–fibered knot in S^3, there exists some open interval (a, b) containing 0 such that for all $r \in (a, b), S^3_r(K)$ admits a taut foliation [LR14]. But they don't say anything about the values of a or b. The L–space conjecture predicts you should get (at least) $r \in (-2g(K)+1, 2g(K)-1)$ (though, actually, for non–fibered knots, you should get taut foliations in $S^3_r(K)$ for all $r \in \mathbb{Q}$).

Prove a quantitative result for non–fibered knots.

Roberts' construction for fibered knots really uses the structure of the fibration to get the foliation. A non–fibered knot doesn't admit a fibration, but it does admit a **broken fibration** – that is, the knot exterior admits a circle–valued Morse function with critical points. Can broken fibrations be used to get quantitative results for non–fibered knots?

Alternatively, try to upgrade the result for fibered knots, and try to get some significant portion of the way to $2g(K)-1$. Siddhi Krishna can do explicitly for positive braid knots.

NB: using Heegaard Floer homology, Adam Levine has an argument that if K is a genus 1 knot, and $n \geq 1$, then $S^3_n(K)$ has LO fundamental group.

Question (Ying Hu): *Does there exist a non-L-space Y, with the property that no co-orientable taut foliation $\mathcal{F}$ admitted by Y has Euler class $e(\mathcal{F}) = 0$?*

Some remarks:

- If Y^3 admits $\mathcal{F}$ is a co-orientable taut foliation (CTF) with Euler class $e(\mathcal{F}) = 0$, then $\pi_1(Y)$ is left-orderable.
- Expected answer is yes. One way to find such examples would be to look at possible Spin^c structures supporting the contact invariant. A variant of the question then is: does there exist a non-L-space, none of whose Spin^c structures with $c_1(\mathfrak{s}) = 0$ support a tight contact structure coming from foliations?

Question (Jonathan Johnson): *Do there exist knots K such that for all $n \geq 2$ the manifolds $\Sigma_n(K)$ are all L-spaces, but $\pi_1(K)$ is not biorderable?*

(1) There are two-bridge knots K which by Peters [Pet09] and Teragaito [Ter14] are known to have the property for all $n \geq 2$ the manifolds $\Sigma_n(K)$ are all L-spaces, which are known to have biorderable knot groups by Johnson [Joh19].
(2) The converse is false. The pretzel knot $P(-5,7,7)$ has biorderable knot group, but $\Sigma_2(K)$ is not an L-space. The converse is still open for fibered knots and alternating knots.

Question (Jacob Caudell): *What is the topological significance of the supported Alexander gradings for an L–space knot?*

Some remarks:

(1) The knot Floer homology $HFK(K)$ of an L-space knot is particularly simple. The highest Alexander grading supported in homology is the genus, and the second highest Alexander grading supported is one less than the genus. What is the topological significance of the other supported Alexander gradings?
(2) Recent work of Tange [Tan19] shows that for knots with a lens space surgery (where K is not a $T(2, 2n+1)$ torus knot) $HFK(K)$ in the third highest Alexander grading is trivial.
(3) For L-space knots which are also links of singularities, the Alexander gradings for which $HFK(K)$ is non-zero are related to the algebraic geometry of the singular curve. See work of Gorsky–Nemethi [GN15], for example.

2. Knots, surfaces, and 4–manifolds

Question (Clayton McDonald): *Does there exist an integer homology 3-sphere Y^3 so that Y^3 embeds smoothly into some integer homology 4-sphere but not into S^4?*

Some remarks:

(1) Many invariants, e.g. d-invariants, do not distinguish between embeddings into S^4 and other integer homology 4-spheres.
(2) Taubes [Tau87] c.f. Daemi [Dae18] may provide techniques sensitive to this question via gauge theory on manifolds with periodic ends. It would be difficult to explicitly construct a manifold satisfying such technical conditions.

Question (Danny Ruberman): *Given an arbitrary integer homology 3-sphere Y^3, does $Y^3 \setminus \mathring{B}^3$ smoothly embed in S^4?*

Some remarks:

(1) This is related to Kirby problem 3.20: Which closed, oriented 3-manifolds embed smoothly into S^4?
(2) By work of Freedman [Fre82], any closed integer homology 3-sphere embeds topologically into S^4.

Question (Kyle Hayden): *Do there exist ribbon disks Σ_1, Σ_2 in B^4 with boundary a knot K so that D_1 and D_2 are isotopic rel boundary but not isotopic rel boundary through ribbon surfaces? (Or similarly for positive genus surfaces?)*

To be precise, does there exist an isotopy rel boundary $\phi_t : (D^2, S^2) \to (B^4, K)$ so that $\phi_t(D^2)$ is ribbon for all but finitely many t, and fails to be ribbon only at the birth or death of critical points of $h|_{\phi_t(D)}$ where h is the radial height function on B^4?

Question (Kyle Hayden): *Are the maps on various Floer theories associated to ribbon surfaces $\Sigma_1, \Sigma_2 \subset B^4$ sensitive to the distinction of isotopic rel boundary vs. isotopic rel boundary through ribbon surfaces?*

In order to make sense of the above question, we must somehow choose decorations of the surfaces.

Question (Jason Joseph and Puttipong Pongtanapaisan): *Are there (non-obvious) additive invariants of 2-knots?*

Some remarks:

(1) For example, the Alexander module is additive under connected sum.
(2) The minimum number of 1-handle surgeries (tubes) required to unknot a surface is not additive.
(3) The minimum number of finger and Whitney moves required to unknot a surface is not additive.
(4) It is unknown whether the ch-index ("crossing hyperbolic," due to Yoshikawa [Yos94]) of a 2-knot is additive. The index measures the minimum number of crossings and bands in a planar diagram of a knotted surface.

(5) It is unknown if the minimum Heegaard genus of a Seifert 3-manifold is additive under connected sum. (In fact, it is not even known whether the connect sum of two nontrivial 2-knots can be the unknotted 2-sphere.)

Question (Peter Feller): *Does there exist a genus-1 cobordism from $T_{3,14}$ to $T_{5,8}$?*

Some remarks:

(1) This is the only pair of torus knots where the existence of a genus–1 cobordism is unknown.
(2) This would be a, "short cobordism," i.e. a cobordism of genus $g_4(T_{3,14}) - g_4(T_{5,8})$
(3) Such a cobordism could possibly come from a complex curve.
(4) The invariant ν^+ fails to obstruct the existence of a genus–1 cobordism. Could $\Sigma_p(-)$ be helpful?
(5) JungHwan Park previously tried a Pin(2) obstruction and could not use this strategy to disprove the existence of a genus-1 cobordism.

Question (Peter Feller): *What is Poincaré triality?*

Trisections yield an order-3 symmetry of 4-manifolds, given by cyclically reordering the pieces of the trisection. Heegaard splittings of 3-manifolds yield a similar order-2 symmetry that induces Poincare duality. What algebraic consequences arise from the order-3 symmetry induced by a trisection?

Question (Clayton McDonald): *Given a knotted $\mathbb{RP}^2$ smoothly embedded in S^4, does it decompose as the connected sum of a knotted 2-sphere and an unknotted $\mathbb{RP}^2$?*

Some remarks:

(1) The affirmative version of this question is known as the *Kinoshita conjecture*.
(2) The relative version is not true: a knotted Möbius band in B^4 need not be the connected sum of a knotted slice disk and an unknotted $\mathbb{RP}^2$. (For example, consider a knot that bounds a Möbius band but not a slice disk, such as $T_{2,2n+1}$ for $n > 0$.)
(3) The link version is not true: A link of $\mathbb{RP}^2$s need not decompose as some connected sum of a link of knotted S^2 and an unlink of $\mathbb{RP}^2$s (Yoshikawa gave the first example [Yos94]).

Question (Aliakbar Daemi): *Let K be a knot in an integer homology sphere $Y \neq S^3$. Define the homotopy Seifert genus and homotopy slice genus of a knot K as*

$$\min\{g(J) \mid J \text{ is homotopic to } K\}$$

and

$$\min\{g_4(J) \mid J \text{ is homotopic to } K\}$$

respectively. Can homotopy Seifert genus be arbitrarily large? Also, what does g_4 mean when $Y \neq S^3$?

A remark: for knots in S^3, these quantities are trivial.

3. Floer homology and low–dimensional topology

Question (Aliakbar Daemi): *Let $T_{p,q}$ be the p-stranded torus knot with q twists. If there exists a concordance from $T_{p,q}$ to some knot K, does there exist a ribbon concordance?*

Some remarks:

(1) The guess is that there should exist knots K concordant to $T_{p,q}$ for which there is not a ribbon concordance.
(2) On the other hand, there is some evidence for a ribbon concordance to exist. Indeed, recall that there is an injective map $H_*(J_1) \to H_*(J_2)$ for a ribbon concordance between knots $J_1 \to J_2$, where H_* represents any of the Floer theories one might like to work with, and that this map is also splittable [DLVVW19]. However, if there is (any) concordance $T_{p,q} \to K$, then the singular instanton Floer homology $I^\#(T_{p,q})$ is a summand of $I^\#(K)$. That is, the concordance behaves as if it is ribbon from the point of view of singular instanton Floer homology. Also: any traceless $SU(2)$ representation of the torus knot extends over the complement of the concordance.

Question (Jennifer Hom): *Gordon's Conjecture: Is ribbon concordance a partial order? That is, if there is a ribbon concordance $K_1 \to K_2$, and one $K_2 \to K_1$, must K_1 and K_2 be the same knot type? If there is a ribbon concordance just one way, K_1 to K_2, is it the case that* $\text{rk } HFK(K_2) > \text{rk } HFK(K_1)$*?*

A remark: note that there are many families of knots with the same knot Floer homology, and see also [Zem19].

Question (Artem Kotelskiy): *If there exists a degree 1 map $f\colon Y \to M$, is it the case that*

$$\text{rk}HF(Y) \geq \text{rk}HF(M)?$$

(Here we use any of the various Floer theories, but for concreteness one can think of Heegaard Floer homology)

(1) There are a number of refinements. In particular, for a null-homotopic knot $K \subset Y$, does surgery on K increase rank? That is,

$$\text{rk } HF(Y_{p/q}(K)) \geq \text{rk } HF(Y)?$$

This is true for knots in S^3. For null-homologous knots, the answer is no.
(2) For instanton homology, one has an inclusion from the chain complex on the right to the chain complex on the left - but it is not clear if it respects the differential.
(3) Given a null-homotopic arc γ in (Y, K) with feet on K, one can perform surgery via deleting a neighborhood of the arc γ and gluing in a rational tangle. (A band sum is a special case of this surgery procedure). In analogy with the above conjecture, it is predicted that:

$$\text{rk}HFK(K_{(\gamma,p/q)}) \geq \text{rk}HFK(K)$$

where HFK is a knot Floer homology theory. An analogous prediction exists for Khovanov homology. If γ is trivial, this is clear by Künneth formula for connected sums.
(4) Artem has tried to use the immersed curve technology to prove this, but it is not clear how to incorporate the null-homotopic condition on the arc.

Question (Tye Lidman): *Recall that the* ***Gromov norm*** *of a three-manifold Y is the infimum of the L^1-norm of singular chains representing the fundamental class. Does there exist an analogous norm for HF? That is, does it make sense to say that a Heegaard Floer class is "large"?*

Some remarks:

(1) Expect to behave well under coverings. Expect nice behavior under cobordisms.
(2) For the contact class, the norm is one. Perhaps working with twisted coefficients or Novikov coefficients, one could retrieve a norm that isn't equal to one on the contact class.
(3) Is this question related to ECH capacity? Recall that the k-th (full) ECH capacity is, roughly speaking, how much symplectic energy is needed to find k independent classes in embedded contact homology.
(4) Another idea is to ask how complicated a given class in HF is relative to cobordism maps.
(5) To make the question more explicit, recall the following: for a homology class $x \in HF^\circ(Y)$, note that for each Heegaard decomposition $\mathcal{H} = (\Sigma, \alpha, \beta)$ of Y, x may be expressed as a sum of tuples $G(\mathcal{H})$ of intersection points of α and β. That is, $x = \sum_{i \in G(\mathcal{H})} x_i$. Here, $\circ$ could be any flavor of HF, but we work with the hat flavor below for simplicity, and mod 2. There is some ambiguity in the choice of set $G(\mathcal{H})$, in that x, viewed as an element of $\widehat{CF}(\Sigma, \alpha, \beta)$ is only well-defined up to homotopy - choose some $G(\mathcal{H})$ for each $\mathcal{H}$ so that $|G(\mathcal{H})|$ is minimal. The question is then to investigate:
$$|x| = \min_{\mathcal{H}} |G(\mathcal{H})|.$$
In particular, is there a 3-manifold Y and a class $x \in \widehat{HF}(Y)$ for which $|x| > 1$?
(6) It may also be of interest to consider this question in the Seiberg-Witten setting.

References

[Bak08] Kenneth L. Baker, *Surgery descriptions and volumes of Berge knots. I. Large volume Berge knots*, J. Knot Theory Ramifications **17** (2008), no. 9, 1077–1097. MR 2457837

[BD15] Jeffrey F. Brock and Nathan M. Dunfield, *Injectivity radii of hyperbolic integer homology 3-spheres*, Geom. Topol. **19** (2015), no. 1, 497–523. MR 3318758

[CD06] Frank Calegari and Nathan M. Dunfield, *Automorphic forms and rational homology 3-spheres*, Geom. Topol. **10** (2006), 295–329. MR 2224458

[Dae18] Aliakbar Daemi, *Chern-Simons Functional and the Homology Cobordism Group*, arXiv e-prints (2018), arXiv:1810.08176.

[DLVVW19] Aliakbar Daemi, Tye Lidman, David Shea Vela-Vick, and C-M Michael Wong, *Obstructions to ribbon homology cobordisms*, preprint, arXiv:1904.09721 (2019).

[Fre82] Michael Hartley Freedman, *The topology of four-dimensional manifolds*, J. Differential Geometry **17** (1982), no. 3, 357–453. MR 679066

[GN15] Eugene Gorsky and András Némethi, *Lattice and Heegaard Floer homologies of algebraic links*, Int. Math. Res. Not. IMRN (2015), no. 23, 12737–12780. MR 3431635

[Joh19] Jonathan Johnson, *Two-Bridge Knots and Residual Torsion-Free Nilpotence*, arXiv (2019).

[LR14] Tao Li and Rachel Roberts, *Taut foliations in knot complements*, Pacific J. Math. **269** (2014), no. 1, 149–168. MR 3233914

[Pet09] Thomas Peters, *On L-spaces and non left-orderable 3-manifold groups*, arXiv (2009).

[Rob01a] Rachel Roberts, *Taut foliations in punctured surface bundles. I*, Proc. London Math. Soc. (3) **82** (2001), no. 3, 747–768.

[Rob01b] ———, *Taut foliations in punctured surface bundles. II*, Proc. London Math. Soc. (3) **83** (2001), no. 2, 443–471.

[Tan19] Motoo Tange, *The third term in lens surgery polynomials*, arXiv (2019).

[Tau87] Clifford Henry Taubes, *Gauge theory on asymptotically periodic* 4*-manifolds*, J. Differential Geom. **25** (1987), no. 3, 363–430. MR 882829

[Ter14] Masakazu Teragaito, *Fourfold cyclic branched covers of genus one two-bridge knots are L-spaces*, Bol. Soc. Mat. Mex. (3) **20** (2014), no. 2, 391–403. MR 3264624

[Yos94] Katsuyuki Yoshikawa, *An enumeration of surfaces in four-space*, Osaka J. Math. **31** (1994), no. 3, 497–522. MR 1309400

[Zem19] Ian Zemke, *Knot Floer homology obstructs ribbon concordance*, Ann. of Math. (2) **190** (2019), no. 3, 931–947. MR 4024565